

Regularization methods for Sliced Inverse Regression

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Joint work with Caroline Bernard-Michel and Laurent Gardes

Outline

- 1 Sliced Inverse Regression (SIR)
- 2 Inverse regression without regularization
- 3 Inverse regression with regularization
- 4 Validation on simulations
- 5 Real data study

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[Li, 1991]

- Infer the conditional distribution of a response r.v. $Y \in \mathbb{R}$ given a predictor $X \in \mathbb{R}^p$.
- When p is large, curse of dimensionality.
- **Sufficient dimension reduction** aims at replacing X by its projection onto a subspace of smaller dimension without loss of information on the distribution of Y given X .
- The **central subspace** is the smallest subspace S such that, conditionally on the projection of X on S , Y and X are independent.

How to estimate a basis of the central subspace?

SIR : Basic principle

Assume $\dim(S) = 1$ for the sake of simplicity, *i.e.* $S = \text{span}(b)$, with $b \in \mathbb{R}^p \implies$ **Single index model** :

$$Y = g(b^t X) + \xi \quad \text{where } \xi \text{ is independent of } X.$$

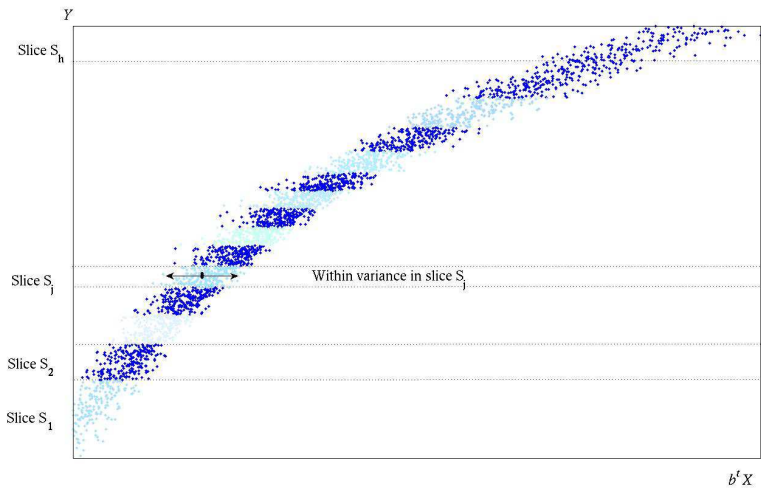
Idea :

- Find the direction b such that $b^t X$ best explains Y .
- Conversely, when Y is fixed, $b^t X$ should not vary.
- Find the direction b minimizing the variations of $b^t X$ given Y .

In practice :

- The range of Y is partitioned into h slices S_j .
- **Minimize the within slice variance of $b^t X$** under the normalization constraint $\text{var}(b^t X) = 1$.
- Equivalent to **maximizing the between slice variance** under the same constraint.

SIR : Illustration



SIR : Estimation procedure

Given a sample $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$, the direction b is estimated by

$$\hat{b} = \underset{b}{\operatorname{argmax}} b^t \hat{\Gamma} b \quad \text{u.c.} \quad b^t \hat{\Sigma} b = 1. \quad (1)$$

where $\hat{\Sigma}$ is the estimated covariance matrix and $\hat{\Gamma}$ is the between slice covariance matrix defined by

$$\hat{\Gamma} = \sum_{j=1}^h \frac{n_j}{n} (\bar{X}_j - \bar{X})(\bar{X}_j - \bar{X})^t, \quad \bar{X}_j = \frac{1}{n_j} \sum_{Y_i \in S_j} X_i,$$

with n_j is proportion of observations in slice S_j . The optimization problem (1) has an explicit solution : \hat{b} is the eigenvector of $\hat{\Sigma}^{-1} \hat{\Gamma}$ associated to its largest eigenvalue.

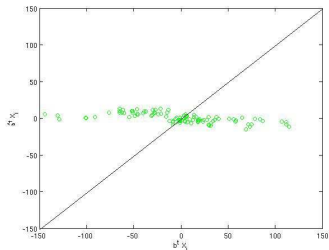
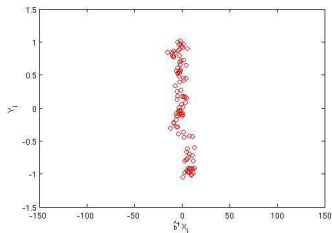
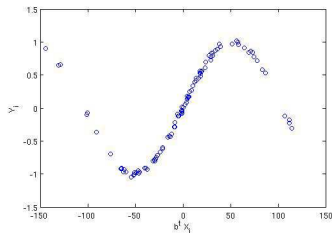
Problem : $\hat{\Sigma}$ can be singular, or at least ill-conditioned, in several situations.

- Since $\text{rank}(\hat{\Sigma}) \leq \min(n - 1, p)$, if $n \leq p$ then $\hat{\Sigma}$ is singular.
- Even when n and p are of the same order, $\hat{\Sigma}$ is ill-conditioned, and its inversion introduces numerical instabilities in the estimation of the central subspace.
- Similar phenomena occur when the coordinates of X are highly correlated.

Experimental set-up.

- A sample $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ of size $n = 100$ where $X_i \in \mathbb{R}^p$ with $p = 50$ and $Y_i \in \mathbb{R}$, for $i = 1, \dots, n$.
 - $X_i \sim \mathcal{N}_p(0, \Sigma)$ with $\Sigma = Q\Delta Q^t$ where
 - $\Delta = \text{diag}(p^\theta, \dots, 2^\theta, 1^\theta)$,
 - Q is a matrix drawn from the uniform distribution on the set of orthogonal matrices.
- \Rightarrow The condition number of Σ is p^θ . (Here, $\theta = 2$).
- $Y_i = g(b^t X_i) + \xi$ where
 - g is the link function $g(t) = \sin(\pi t/2)$,
 - b is the true direction $b = 5^{-1/2}Q(1, 1, 1, 1, 1, 0, \dots, 0)^t$,
 - $\xi \sim \mathcal{N}_1(0, 9.10^{-4})$

SIR : Numerical experiment (2/2)



Blue : Projections $b^t X_i$ on the true direction b versus Y_i ,

Red : Projections $\hat{b}^t X_i$ on the estimated direction \hat{b} versus Y_i ,

Green : $b^t X_i$ versus $\hat{b}^t X_i$.

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Single-index inverse regression model

Model introduced in [Cook, 2007].

$$X = \mu + c(Y)Vb + \varepsilon, \quad (2)$$

where

- μ and b are non-random \mathbb{R}^p - vectors,
- $\varepsilon \sim \mathcal{N}_p(0, V)$, independent of Y ,
- $c: \mathbb{R} \rightarrow \mathbb{R}$ is a nonrandom coordinate function.

Consequence : The conditional expectation of $X - \mu$ given Y is a degenerated random vector located in the direction Vb .

Maximum Likelihood estimation (1/3)

- **Projection estimator of the coordinate function.** $c(\cdot)$ is expanded as a linear combination of h basis functions $s_j(\cdot)$,

$$c(\cdot) = \sum_{j=1}^h c_j s_j(\cdot) = s^t(\cdot) c,$$

where $c = (c_1, \dots, c_h)^t$ is unknown and $s(\cdot) = (s_1(\cdot), \dots, s_h(\cdot))^t$. Model (2) can be rewritten as

$$X = \mu + s^t(Y) c V b + \varepsilon, \quad \varepsilon \sim \mathcal{N}_p(0, V),$$

- Definition : **Signal to Noise Ratio in the direction b .**

$$\rho = \frac{b^t \Sigma b - b^t V b}{b^t V b},$$

where $\Sigma = \text{cov}(X)$.

Notations

- W : the $h \times h$ empirical covariance matrix of $s(Y)$ defined by

$$W = \frac{1}{n} \sum_{i=1}^n (s(Y_i) - \bar{s})(s(Y_i) - \bar{s})^t \quad \text{with} \quad \bar{s} = \frac{1}{n} \sum_{i=1}^n s(Y_i).$$

- M : the $h \times p$ matrix defined by

$$M = \frac{1}{n} \sum_{i=1}^n (s(Y_i) - \bar{s})(X_i - \bar{X})^t,$$

Maximum Likelihood estimation (3/3)

If W and $\hat{\Sigma}$ are regular, then the ML estimators are :

- **Direction** : \hat{b} is the eigenvector associated to the largest eigenvalue $\hat{\lambda}$ of $\hat{\Sigma}^{-1}M^tW^{-1}M$,
- **Coordinate** : $\hat{c} = W^{-1}M\hat{b}/\hat{b}^t\hat{V}\hat{b}$,
- **Location parameter** : $\hat{\mu} = \bar{X} - \bar{s}^t\hat{c}\hat{V}\hat{b}$,
- **Covariance matrix** : $\hat{V} = \hat{\Sigma} - \hat{\lambda}\hat{\Sigma}\hat{b}\hat{b}^t\hat{\Sigma}/\hat{b}^t\hat{\Sigma}\hat{b}$,
- **Signal to Noise Ratio** : $\hat{\rho} = \hat{\lambda}/(1 - \hat{\lambda})$.

The inversion of $\hat{\Sigma}$ is still necessary.

SIR : A particular case

In the particular case of **piecewise constant basis functions**

$$s_j(.) = \mathbb{I}\{. \in S_j\}, \quad j = 1, \dots, h,$$

standard calculations show that

$$M^t W^{-1} M = \hat{\Gamma}$$

and thus the ML estimator \hat{b} of b is the eigenvector associated to the largest eigenvalue of $\hat{\Sigma}^{-1} \hat{\Gamma}$.

\Rightarrow SIR method.

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Introduction of a prior information on the projection of X on b appearing in the inverse regression model

$$(1 + \rho)^{-1/2} (s(Y) - \bar{s})^t c b \sim \mathcal{N}(0, \Omega).$$

- $(1 + \rho)^{-1/2}$ is introduced for normalization purposes, permitting to preserve the interpretation of the eigenvalue in terms of signal to noise ratio.
- Ω describes which directions in \mathbb{R}^p are the most likely to contain b .

Gaussian regularized estimators

If W and $\Omega\hat{\Sigma} + I_p$ are regular, the ML estimators are

- **Direction** : \hat{b} is the eigenvector associated to the largest eigenvalue $\hat{\lambda}$ of $(\Omega\hat{\Sigma} + I_p)^{-1}\Omega M^t W^{-1}M$,
- **Coordinate** : $\hat{c} = W^{-1}M\hat{b}/((1 + \eta(\hat{b}))\hat{b}^t\hat{V}\hat{b})$, with $\eta(\hat{b}) = \hat{b}^t\Omega^{-1}\hat{b}/\hat{b}^t\hat{\Sigma}\hat{b}$,
- $\hat{\mu}$, \hat{V} and $\hat{\rho}$ are unchanged.

\Rightarrow The inversion of $\hat{\Sigma}$ is replaced by the inversion of $\Omega\hat{\Sigma} + I_p$.

\Rightarrow For a properly chosen prior matrix Ω , the numerical instabilities in the estimation of b disappear.

Gaussian regularized SIR (1/2)

GRSIR : In the particular case of piecewise constant basis functions, the ML estimator \hat{b} of b is the eigenvector associated to the largest eigenvalue of $(\Omega\hat{\Sigma} + I_p)^{-1}\Omega\hat{\Gamma}$.

Links with existing methods

- Ridge [Zhong et al, 2005] : $\Omega = \tau^{-1}I_p$. No privileged direction for b in \mathbb{R}^p . $\tau > 0$ is the regularization parameter.
- PCA+SIR [Chiaromonte et al, 2002] :

$$\Omega = \sum_{j=1}^d \frac{1}{\hat{\delta}_j} \hat{q}_j \hat{q}_j^t,$$

where $d \in \{1, \dots, p\}$ is fixed, $\hat{\delta}_1 \geq \dots \geq \hat{\delta}_d$ are the d largest eigenvalues of $\hat{\Sigma}$ and $\hat{q}_1, \dots, \hat{q}_d$ are the associated eigenvectors.

Three new methods

- PCA+ridge :

$$\Omega = \frac{1}{\tau} \sum_{j=1}^d \hat{q}_j \hat{q}_j^t.$$

No privileged direction in the d -dimensional eigenspace.

- Tikhonov : $\Omega = \tau^{-1} \hat{\Sigma}$. Directions with large variance are most likely.
- PCA+Tikhonov :

$$\Omega = \frac{1}{\tau} \sum_{j=1}^d \hat{\delta}_j \hat{q}_j \hat{q}_j^t.$$

In the d -dimensional eigenspace, directions with large variance are most likely.

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Experimental set-up : Same as previously.

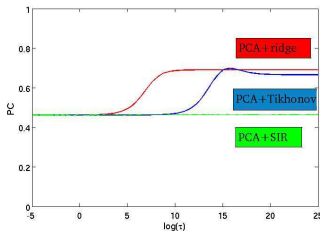
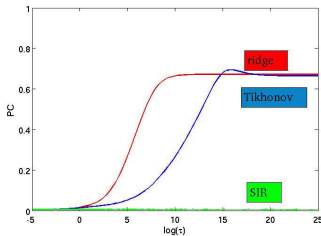
Proximity criterion between the true direction b and the estimated ones $\hat{b}^{(r)}$ on $N = 100$ replications :

$$\text{PC} = \frac{1}{N} \sum_{r=1}^N (b^t \hat{b}^{(r)})^2$$

- $0 \leq \text{PC} \leq 1$,
- a value close to 0 implies a low proximity : The $\hat{b}^{(r)}$ are nearly orthogonal to b ,
- a value close to 1 implies a high proximity : The $\hat{b}^{(r)}$ are approximatively collinear with b .

Influence of the regularization parameter

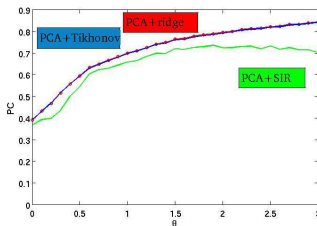
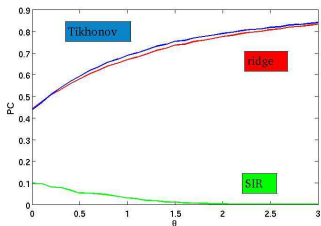
$\log \tau$ versus PC. The “cut-off” dimension and the condition number are fixed ($d = 20$ and $\theta = 2$).



- **Ridge** and **Tikhonov** : significant improvement if τ is large,
- **PCA+SIR** : reasonable results compared to **SIR**,
- **PCA+ridge** and **PCA+Tikhonov** : small sensitivity to τ .

Sensitivity with respect to the condition number of the covariance matrix

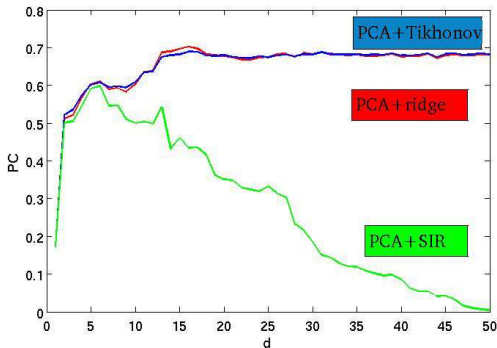
θ versus PC. The “cut-off” dimension is fixed to $d = 20$. The optimal regularization parameter is used for each value of θ .



- Only **SIR** is very sensitive to the ill-conditioning,
- **ridge** and **Tikhonov** : similar results,
- **PCA+ridge** and **PCA+Tikhonov** : similar results.

Sensitivity with respect to the “cut-off” dimension

d versus PC. The condition number is fixed ($\theta = 2$) The optimal regularization parameter is used for each value of d .



- **PCA+SIR** : very sensitive to d .
- **PCA+ridge** and **PCA+Tikhonov** : stable as d increases.

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Estimation of Mars surface physical properties from hyperspectral images

Context :

- Observation of the south pole of Mars at the end of summer, collected during orbit 61 by the French imaging spectrometer OMEGA on board Mars Express Mission.
- 3D image : On each pixel, a spectra containing $p = 184$ wavelengths is recorded.
- This portion of Mars mainly contains water ice, CO_2 and dust.

Goal : For each spectra $X \in \mathbb{R}^p$, estimate the corresponding physical parameter $Y \in \mathbb{R}$ (grain size of CO_2).

An inverse problem

Forward problem.

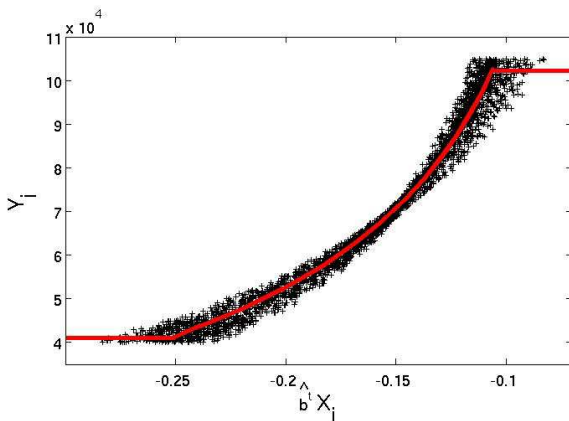
- Physical modeling of individual spectra with a surface reflectance model.
- Starting from a physical parameter Y , simulate $X = F(Y)$.
- Generation of $n = 12,000$ synthetic spectra with the corresponding parameters.

⇒ Learning database.

Inverse problem.

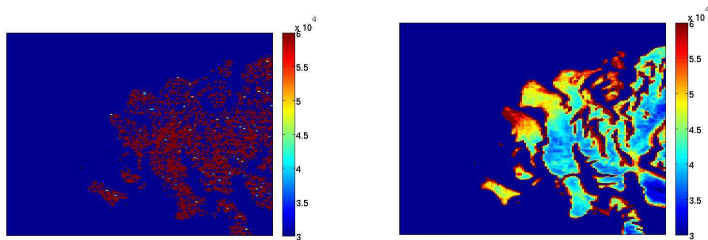
- Estimate the functional relationship $Y = G(X)$.
- Dimension reduction assumption $G(X) = g(b^t X)$.
- b is estimated by SIR/GRSIR, g is estimated by a nonparametric one-dimensional regression.

Estimated functional relationship



Functional relationship between reduced spectra $\hat{b}^t X$ on the first GRSIR (PCA+ridge prior) direction and Y , the grain size of CO_2 .

Estimated CO₂ maps



Grain size of CO₂ estimated by SIR (left) and GRSIR (right) on an hyperspectral image observed on Mars during orbit 61.

- [Li, 1991] Li, K.C. (1991). Sliced inverse regression for dimension reduction. *Journal of the American Statistical Association*, **86**, 316–327.
- [Cook, 2007]. Cook, R.D. (2007). Fisher lecture : Dimension reduction in regression. *Statistical Science*, **22**(1), 1–26.
- [Zhong et al, 2005]. Zhong, W., Zeng, P., Ma, P., Liu, J.S. and Zhu, Y. (2005). RSIR : Regularized Sliced Inverse Regression for motif discovery. *Bioinformatics*, **21**(22), 4169–4175.
- [Chiaromonte et al, 2002]. Chiaromonte, F. and Martinelli, J. (2002). Dimension reduction strategies for analyzing global gene expression data with a response. *Mathematical Biosciences*, **176**, 123–144.

References

- R. Coudret, S. Girard & J. Saracco. A new sliced inverse regression method for multivariate response, *Computational Statistics and Data Analysis*, to appear, 2014.
- M. Chavent, S. Girard, V. Kuentz, B. Lique, T.M.N. Nguyen & J. Saracco. A sliced inverse regression approach for data stream, *Computational Statistics*, to appear, 2014.
- C. Bernard-Michel, S. Douté, M. Fauvel, L. Gardes & S. Girard. Retrieval of Mars surface physical properties from OMEGA hyperspectral images using Regularized Sliced Inverse Regression, *Journal of Geophysical Research - Planets*, 114, E06005, 2009.
- C. Bernard-Michel, L. Gardes & S. Girard. A Note on Sliced Inverse Regression with Regularizations, *Biometrics*, 64, 982–986, 2008.
- C. Bernard-Michel, L. Gardes & S. Girard. *Gaussian Regularized Sliced Inverse Regression*, *Statistics and Computing*, 19, 85–98, 2009.
- A. Gannoun, S. Girard, C. Guinot & J. Saracco. Sliced Inverse Regression in reference curves estimation, *Computational Statistics and Data Analysis*, 46, 103–122, 2004.